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Modelling Temperature Using CARMA Processes with Stochastic Speed of Mean Reversion for Temperature Insurance Pricing

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Abstract

In this paper, we present a continuous time autoregressive moving average (CARMA) model with stochastic speed of mean reversion. This model allows the mean reversion rates to behave stochastically and governed by an Ornstein-Uhlenbeck process. We provide closed-form solution to the CARMA with stochastic speed of mean reversion and formulate the price of temperature insurance using spot-forward relationship framework. We demonstrate the insurance pricing based on the cumulative average temperatures (CAT) index by simulating the temperature variations. We found that our proposed model may explain the temperature evolution well and the price of CAT-based index insurance looks reasonable.

Keywords: stochastic process; continuous autoregressive moving average processes; mean reversion; temperature model; temperature insurance.

1 Introduction

Continuous time autoregressive moving average (CARMA) is a general class of stationary processes which has been widely used in many fields including physics and engineering for many years. In finance, the process is popularly employed in modelling the spot price dynamics of many financial assets. A work by Andresen et. al [1] has used CARMA processes to explain the evolution of the short and formed interest rates. Garcia et. al [11] have theoretically studied the properties and estimation of CARMA processes with the application in energy market. In energy related markets for example temperature, the CARMA processes have been empirically studied by Benth and Săltyte [6] for the purpose of energy derivatives pricing. These studies motivate the use of CARMA model in weather index insurance by Taib and Darus [14]. In freight market, Benth et al. [5] has proposed the CARMA process to model the spot freight rates. A numerical investigation of the CARMA has been done by Engan [10] in pricing forward and options contracts.

Lévy driven CARMA processes is defined as $Y(t) = \mathbf{b}'\mathbf{X}(t)$, where $\mathbf{X}(t)$ be the solution of the stochastic differential equation

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + \mathbf{e}_p\sigma dL(t),$$

where A is a $p \times p$ matrix containing p mean reversion parameters. The mean reversion is related to the stationary property of the process where the process is assumed to revert back to their long term average. By setting $p = 1$, we have $A = \alpha$. This lead us to the Ornstein-Uhlenbeck (OU) process as a subclass of CARMA defined by (see Benth and Săltyte [6])

$$dX(t) = -\alpha X(t)dt + \sigma dL(t). \tag{1}$$

This process has been applied in finance particularly in option pricing (see Ibrahim [13]) and also in insurance (see Dufresne [9]).

The speed of mean reversion parameters are normally considered as constant. However, we may from the previous works of Barlow et al. [2] see that the constancy of the mean reversion rate is problematic where their study found the uncertainty in the estimation of the speed of mean reversion. In addition, the study by Zapranis and Alexandridis [15] suggests that the mean reversion rate is not constant and change over time. Furthermore, Benth and Khedher [4] have proposed the stochastic process which allow the speed of mean reversion to be stochastic. The findings later have been applied in the calibration of futures contracts on temperatures index. Following Benth and Khedher [4], we consider the stochastic speed of mean reversion in this paper, by allowing matrix A be $A(t)$, and the element $\alpha_i(t)$ in the matrix is the stochastic mean reversion rate. To the best of our knowledge, this is the first study considering the stochastic speed of mean reversion for CARMA processes.

One of the issues when allowing mean reversion parameter of the CARMA be stochastic is the stationary property of the process still hold. Stationarity solution to the stochastic differential equation is required to ensure the mean and variance do not change over time. Previous study in determining the necessary and sufficient conditions for the existence of the strictly stationary solutions of Lévy-driven CARMA process was done by Brockwell and Lindner [8]. Similar investigation has also been done by Gushchin and Kuchler [12]. Our study found that stationary property still hold for the CARMA processes with stochastic speed of mean reversion.

The paper is organised as follows: Section 2 discusses the main part of the study, that is the introduction of the CARMA processes with the stochastic speed of mean reversion. In Section 3, we provide empirical analysis of our temperature data. Section 4 is devoted to the simulation procedures for CARMA processes with stochastic speed of mean reversion for temperatures. Finally, Section 5 concludes our paper.

2 Lévy-Driven CARMA Process with Stochastic Speed of Mean Reversion

Let $L(t)$ be a Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. For $0 \leq q < p$, we denote $\mathbf{b}$ as a vector which contains elements b_j, $j = 0, \ldots, p-1$ with $b_j = 0$ and $b_q = 1$ for $q < j < p$. The CARMA(p, q) process is defined as

$$Y(t) = \mathbf{b}'\mathbf{X}(t), \tag{2}$$

where $\mathbf{X}(t) \in \mathbb{R}^p$ satisfies the following stochastic differential equation

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p \sigma dL(t). \tag{3}$$

Here, parameter σ is a positive constant and $\mathbf{e}_k (k = 1, \ldots, p)$ is the kth unit vector in $\mathbb{R}^p$. The $p \times p$ matrix A contains parameter $\alpha_i (i = 1, \ldots, p)$ which is given by,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{bmatrix}. \tag{4}$$

Note that α_i is referred to the speed of mean reversion. For $p = 1$, we have $A = \alpha_1 = \alpha$ and (3) can be re-expressed as

$$dX(t) = -\alpha X(t)dt + \sigma dL(t). \tag{5}$$

Equation (5) is popularly known as the OU process. The solution of (3) for $s \geq t$ yielding

$$\mathbf{X}(s) = e^{A(s-t)}\mathbf{X}(t) + \int_t^s e^{A(s-u)}\mathbf{e}_p \sigma dL(u). \tag{6}$$

Hence, the CARMA process (2) is given as

$$Y(t) = \mathbf{b}'e^{A(s-t)}\mathbf{X}(s) + \int_s^t \mathbf{b}'e^{A(t-u)}\mathbf{e}_p \sigma dL(u). \tag{7}$$

Equations (2) and (3) can also be represented as (see Brockwell [7])

$$a(D)Y(t) = b(D)DL(t), \tag{8}$$

where D is the differentiation with respect to t, while $a(\cdot)$ and $b(\cdot)$ are the polynomials

$$\begin{aligned} a(z) &= z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p, \\ b(z) &= b_0 + b_1 z + \cdots + b_p z^p. \end{aligned} \tag{9}$$

According to Brockwell [7], the CARMA process is stationary if and only if the real parts of the eigenvalues of A are negative, that is $\text{Re}(\lambda_i) < 0$ for $i = 1, \ldots, p$, then the roots of the polynomials (9) are distinct.

Further, we redefine $\mathbf{X}(t)$ to be the stochastic process with stochastic speed of mean reversion given as

$$d\mathbf{X}(t) = A(t)\mathbf{X}(t)\,dt + \mathbf{e}_p \sigma dL(t). \tag{10}$$

Note that A is now changed to $A(t)$ to represent the matrix which containing the stochastic mean reversion rates. The following proposition is devoted to the explicit solution of (10). Remark that $A(t)$ is measurable and adapted process satisfying the integrability conditions.

Proposition 2.1. *Suppose that $A(t)$ is integrable on $[0, T]$ for a given $T < \infty$, then*

$$\mathbb{E}\left[\exp\left(\int_0^T A(u)du\right)\right] < \infty.$$

For $t \leq T$, the $\mathcal{F}_t$-adapted process

$$\mathbf{X}(t) = \exp\left(\int_0^t A(u)du\right)\mathbf{X}(0) + \exp\int_0^t A(u)du \int_0^t \mathbf{e}_p\sigma\exp\left(-\int_0^s A(u)du\right)dL(s),$$

is the solution of (10).

Proof. By applying Itô formula, we get

$$\begin{aligned} d\left(\mathbf{X}(t)\exp\left(\int_0^t A(u)du\right)\right) =& - A(t)\mathbf{X}(t)\exp\left(-\int_0^t A(u)du\right)dt + \exp\left(-\int_0^t A(u)du\right)dL(t) \\ =&\mathbf{e}_p\sigma\exp\left(-\int_0^t A(u)du\right)dL(t). \end{aligned}$$

Thus, we have

$$\begin{aligned} d\left(\mathbf{X}(t)\exp\left(-\int_0^t A(u)du\right)\right) &= \mathbf{e}_p\sigma\exp\left(-\int_0^t A(u)du\right)dL(t) \\ \mathbf{X}(t)\exp\left(-\int_0^t A(u)du\right) &= \mathbf{X}(0) + \int_0^t \mathbf{e}_p\,\sigma\exp\left(-\int_0^s A(u)du\right)dL(s). \end{aligned}$$

The proposition follows. □

Hence, the new CARMA process is given as

$$Y(t) = \mathbf{b}'\exp\left(\int_0^t A(u)du\right)\mathbf{X}(0) + \exp\left(\int_0^t A(u)du\right)\int_0^t \mathbf{b}'\mathbf{e}_p\,\sigma\exp\left(-\int_0^s A(u)du\right)dL(s). \tag{11}$$

Here, $A(t)$ is the $p \times p$-matrix given by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p(t) & -\alpha_{p-1}(t) & -\alpha_{p-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix}, \tag{12}$$

where $\alpha_k(t)$ for $k = 1, \dots, p$ are the stochastic speed of mean reversions. Following Benth and Khedher [4], we define $\alpha_k(t)$ as stationary Lévy processes given by

$$d\alpha_k(t) = \beta(\mu - \alpha_k(t))dt + \eta dZ(t), \tag{13}$$

where $\beta > 0$ and $\eta > 0$ are two constants representing the mean reversion rate and volatility of speed of mean reversion respectively. The value μ is referred to a long term mean level. Such process is driven by subordinator $Z(t)$, that is the Lêvy process with increasing path. The next proposition derive the explicit solution of (13).

Proposition 2.2. *For $s \leq t$, the $\mathcal{F}_t$-adapted process*

$$\alpha_k(t) = \mu\left(1 - \exp\left(-\beta t\right)\right) + \exp\left(-\beta t\right)\alpha_k(s) + \eta \int_s^t \exp\left(-\beta\right)\left(t-u\right)dZ(u),$$

is the solution of (13).

Proof. Let μ be a long-term mean, by changing of variable we have

$$U(t) = \alpha_k(t) - \mu.$$

Express $U(t)$ in form of

$$dU(t) = d\alpha_k(t) = -\beta U(t)dt + \eta dZ(t). \tag{14}$$

Equation 14 has drift toward zero at an exponential $-\beta$, and by changing the variables, we have

$$U(t) = \exp(-\beta t)R(t) \Leftrightarrow R(t) = \exp(\beta t)U(t).$$

Applying the Itô formula, we obtain

$$\begin{aligned} d\left(\exp(\beta t)U(t)\right) &= \beta \exp(\beta t)U(t)dt + \exp(\beta t)dU(t) \\ &= \eta \exp(\beta t)dZ(t). \end{aligned} \tag{15}$$

The solution for $R(t)$ is immediately obtained by Itô-integrating for $t \geq 0$

$$R(t) = R(s) + \eta \int_s^t \exp\left(\beta u\right)dZ(u),$$

and by reverting the change, we get

$$\begin{aligned} U(t) &= \exp(-\beta t)R(t) \\ &= \exp(-\beta t)U(t) + \eta \exp(-\beta t)\int_s^t \exp(\beta(t-u))dZ(u), \end{aligned}$$

and finally we obtain

$$\begin{aligned} \alpha_k(t) &= U(t) + \mu \\ &= \mu\left(1 - \exp\left(-\beta t\right)\right) + \exp\left(-\beta t\right)\alpha_k(s) + \eta \int_s^t \exp\left(-\beta\right)\left(t-u\right) dZ(u). \end{aligned} \quad (16)$$

□

If $\beta > 0$, then $\lim_{t\to\infty} \int_s^t e^{-\beta\, u} dZ(u)$ exists a.s finite random variable, which can be stated as $\int_t^\infty e^{-\beta\, u} dZ(u)$. But, if $\beta = 0$, then $\alpha_k(t) = \alpha_k(0) + \eta\, Z(t)$ where $t \geq 0$. This $\alpha_k(t)$ process has a positive increment when Z has positive increment. If μ is positive then the mean $\alpha_k(t)$ is also positive, otherwise the negative value of α will give non-stationary behaviour. Hence, when $p = 1$ we have $A(t) = \alpha_1(t) = \alpha(t)$. The polynomial $A(t)$ can be restated as

$$\begin{aligned} a_t(z) &= z^p + \alpha_1(t) z^{p-1} + \cdots + \alpha_p(t), \\ b_t(z) &= b_0(t) + b_1(t) z + \cdots + b_q(t) z^q. \end{aligned} \quad (17)$$

The generalized CARMA process with stochastic speed of mean reversion is stationary if and only if the real parts of the eigenvalues for every time t, $A(t)$ are negative, that is $\text{Re}(\lambda_i)(t) < 0$ for $i = 1, \ldots, p$, then the roots of the polynomials (17) are distinct.

3 Empirical Analysis of Temperatures Data

We use data of the Kota Bharu, Malaysia daily average temperatures (DATs) which are recorded in degrees of Celsius. The data from 1 January 1997 until 6 September 2016 are obtained from Malaysian Meteorological Department. To synchronise the length of data in each year to 365 days, we have removed records on 29 February in each leap year, leave us with number of 7095 observations. We have checked and found no missing data in the time series. Figure 1 illustrates the time series of DATs for the last 5 years.

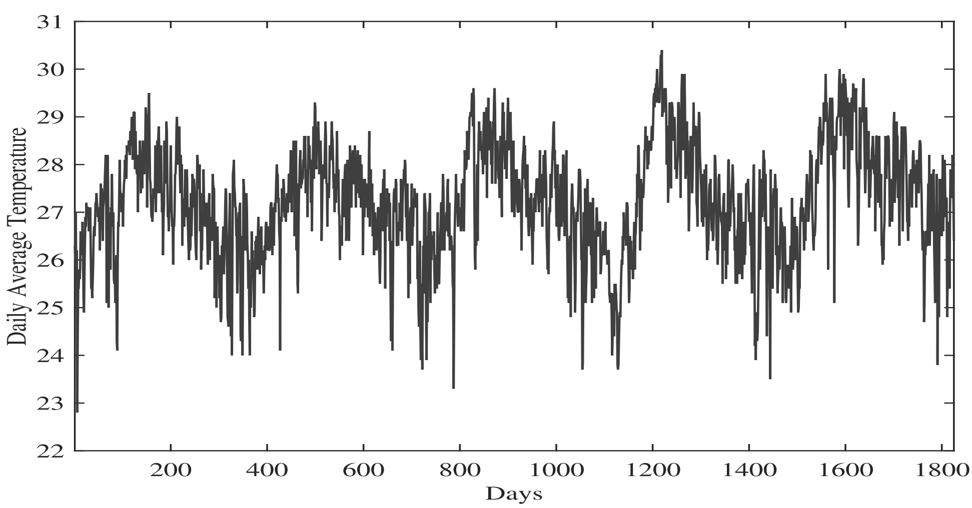

Figure 1: Time series of DATs for Kota Bharu.

The descriptive statistics of DATs is presented in Table 1. We may clearly see the seasonal pattern of DATs in Kota Bharu where temperatures vary in the range of $(22.8, 40.9)$. We found that the average of the temperature is 27.3, while the standard deviation is equal to 1.1. The value of skewness equals -0.2449 which indicates the left skewness of temperature and kurtosis equals 3.306 almost likely to the normal distribution. The histogram plot of the temperature distribution in Figure 2 however gives sign of deviation from normality.

Table 1: Descriptive statistics of DATs.

Min	Max	Mean	Std Dev	Skewness	Kurtosis
22.8	40.9	27.3	1.1	-0.24491	3.306

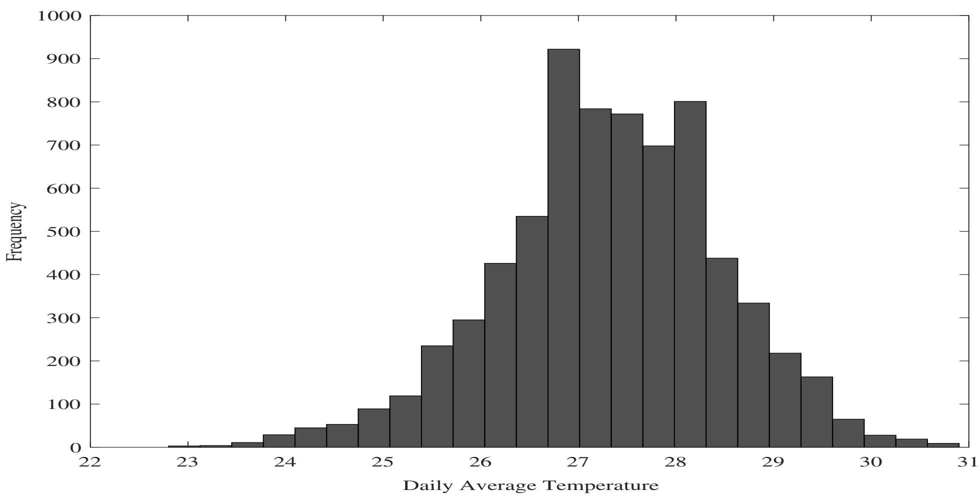

Figure 2: Histogram of DATs.

3.1 Trend and Seasonal Component

We use a seasonal function given as

$$\Lambda(t) = a_0 + a_1 t + a_2 \sin\left(\frac{2\pi\,(t - a_3)}{365}\right), \tag{18}$$

where constant a_0 is the average level of temperature and a_1 is the slope of a linear trend function. While the constants a_2 and a_3 shows the amplitude of the mean and the phase angle, respectively.

Figure 3 illustrates the DATs fitted with seasonal function (18) for the last 5 years. In Figure 4, we plot the autocorrelation function of DATs which shows a clear seasonal effect in the plot.

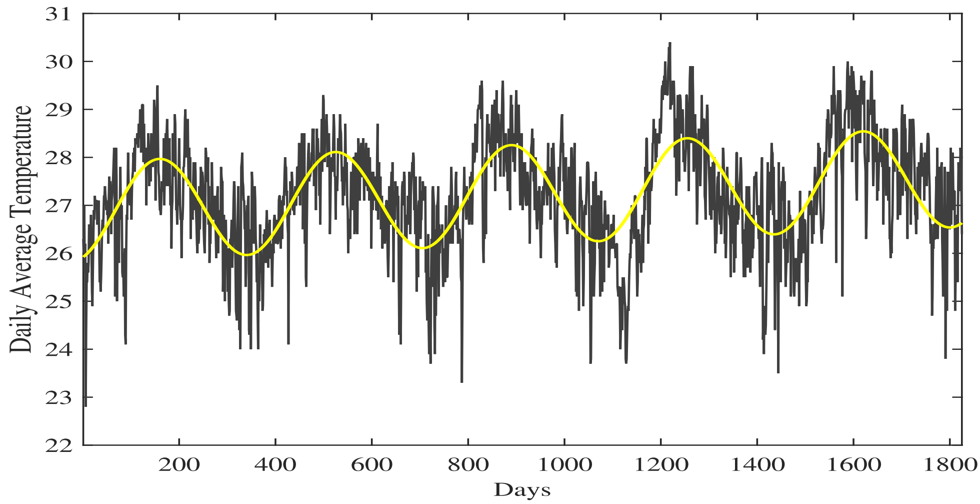

Figure 3: DATs of Kota Bharu together with fitted seasonal function.

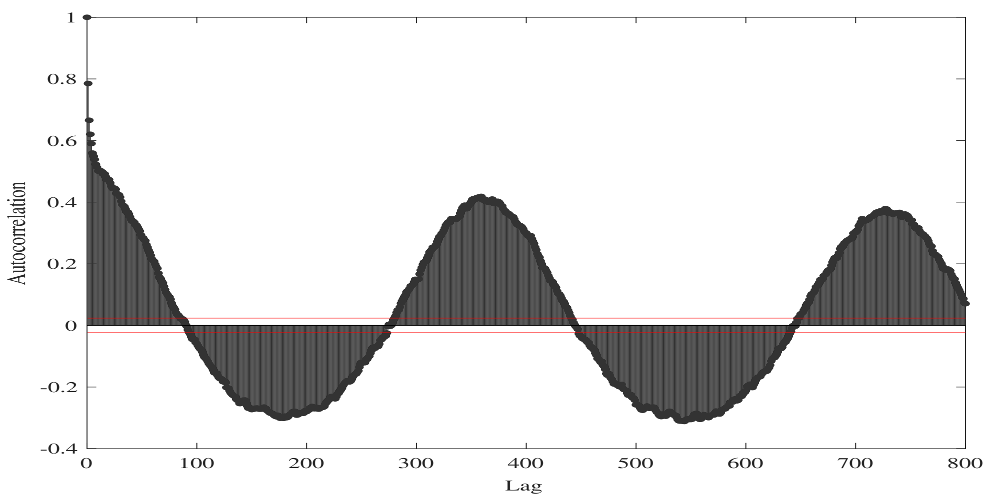

Figure 4: The autocorrelation function of DATs for Kota Bharu.

Table 2 reports parameters which are estimated from the least square fitting of seasonal function to the DATs data [1]. The $a_1 = 0.00001$ indicates that the trend of temperatures series is increasing at a slow rate with a very small P-value of 0.0000. The seasonal variations are small by looking at the amplitude a_2 equals 0.9926.

Table 2: Estimates from seasonal function fitting.

a_0	a_1	a_2	a_3
27.3544	0.00001	0.9926	65.6056

[1]We use *nlinfit* function in Matlab for least square estimation procedure.

The temperature are detrended and deseasonalized by subtracting the estimated $\Lambda(t)$ from our observed temperature $T(t)$, that is

$$Y(t) = T(t) - \Lambda(t). \tag{19}$$

Based on the partial autocorrelation (PACF) plot in Figure 5, it is suggested to use CARMA$(3, 0)$ or simply CAR(3) for the deseasonalized temperatures dynamics.

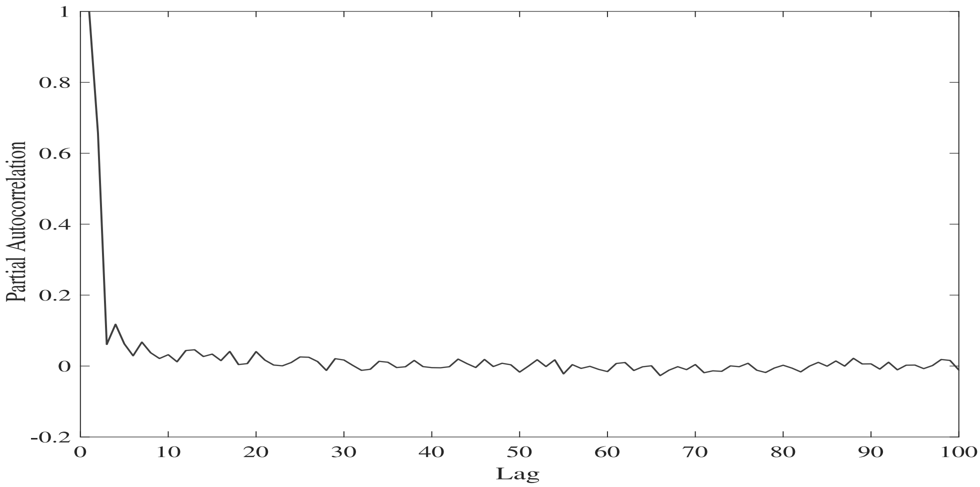

Figure 5: The PACF of the detrended and deseasonalized DATs of Kota Bharu.

3.2 CAR(3) Model

We may represent the CAR(3) on a daily scale $Y(t_n) \approx y_n$ in discrete form as AR(3) given by

$$y_n = \beta_1 y_{n-1} + \beta_2 y_{n-2} + \beta_3 y_{n-3} + \epsilon_n,$$

where β_1, β_2 and β_3 are constants and ϵ_n are $i.i.d$ residuals. All the parameters are obtained from fitting the deseasonalized DATs to AR(3) model which are reported in Table 3. Since our model is in continuous time, we apply the following relation

$$\begin{aligned} 3 - \alpha_1 &= \beta_1, \\ 2\alpha_1 - \alpha_2 - 3 &= \beta_2, \\ \alpha_2 + 1 - (\alpha_1 + \alpha_3) &= \beta_3, \end{aligned} \tag{20}$$

to obtain the estimates for CAR(3) (refer to Benth and Šaltyte [6]) as in Table 4.

Table 3: Regression parameters of AR(3).

β_1	β_2	β_3
-0.6095	0.0115	-0.1176

Table 4: Fitted regression parameters of CAR(3).

α_1	α_2	α_3
3.6095	4.2075	1.7156

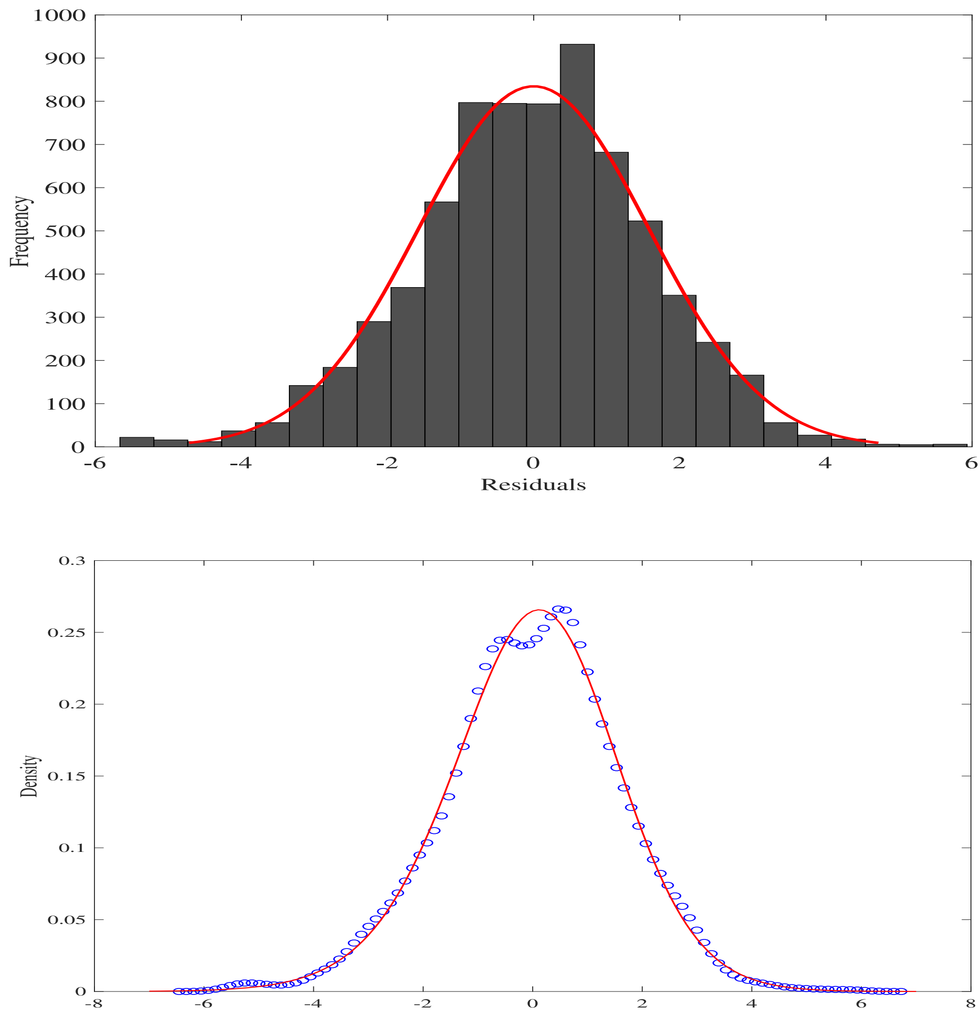

Figure 6: Plot of (top) histogram of residuals fitted with normal distribution and (bottom) density of residuals fitted with NIG.

The eigenvalues of matrix A are $\lambda_1 = -1.8052$ and $\lambda_{2,3} = -0.9022 \pm 0.3694\mathrm{i}$. Since all the eigenvalues have negative real parts, then the stationary condition of the CAR(3) process holds. By looking at Figure 6, it seems that histogram of residuals after removing the autoregressive effect follows the normal distribution but the reported values of skewness equal -0.1737 and kurtosis equals 3.4139 reject the normality. The standard deviation of residuals is 1.5725. Hence, we suggest to use the normal inverse Gaussian (NIG) distribution to model the residuals.

The NIG distribution was introduced by Barndorff-Nielsen [3] in order to model the returns of financial time series with four parameters: α, β, δ and μ. Its probability density function is defined as

$$f(x;\alpha,\beta,\delta,\mu) = k\exp(\beta(x-\mu))\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}},$$

where α represents the heaviness of the tails, while β and δ are the skewness and scale parameters, respectively. The parameter for the location of the distribution is represented by μ. Here, $k = \delta\alpha\exp(\delta\sqrt{\alpha^2-\beta^2})/\pi$ and $K_1(x)$ is the modified Bessel function of the third kind with index 1. The density of residuals fitted with NIG are plotted in Figure 6 and the estimated parameters of NIG are reported in Table 5.

Table 5: Estimated parameters of NIG distribution.

α	β	δ	μ
1.9157	0.4537	3.1315	0.7604

4 Simulation of the Daily Average Temperatures

Our aim here is to simulate paths of DATs using CAR(3) with stochastic speed of mean reversion model. From (19), temperature at time $t \geq 0$ can be obtained by adding the seasonal mean function to the CAR(3) process, that is $T(t) = \Lambda(t) + Y(t)$. However, this is not straightforward since $Y(t)$ contains stochastic speed of mean reversion inside matrix $A(t)$. Thus, in a step-by-step procedure, we have to simulate the mean reversion rates using (13) and the $\alpha_k(t)$ for $k = 1, ..., 3$ at each time t are used to form the matrix A. Then, we simulate $\mathbf{X}(t)$ for $t = 1, ..., 365$ in a usual way.

We take our estimates in Table 4 as the initial values for the stochastic speed of mean reversion, $a_k(0)$. We simulate a path of daily $\alpha_k(t)$ for $t = 1, ..., 365$, which covers for one year period. To have the NIG distributed returns, we let $L = B$, a Brownian motion. Then, the subordinator Z follows the inverse Gaussian family, in which the process $\alpha_k(t)$ becomes stationary. Eq. (16) can be discretized and approximated via Euler scheme given as

$$\alpha_k(t+1) = \mu\left(1-e^{-\beta}\right) + e^{-\beta}\alpha(t) + \eta\, e^{-\beta}\Delta L(s), \tag{21}$$

where $L \sim \mathrm{NIG}(\alpha,\beta,\delta,\mu)$. Note that the value of parameters β, μ and η must be positive. It is hard to estimate such parameters since $\alpha_k(t)$ are not observable. However, to our best (just for illustration), we let $\beta = 0.9$, $\mu = 0.55$ and $\eta = 0.85$, and $p = 3$ which look reasonable. We have three different mean reversion rates inside matrix $A(t)$: $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$. We also assume that the process $Z(t)$ are independent of $L(t)$. The simulated paths of $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ are plotted in Figure 7.

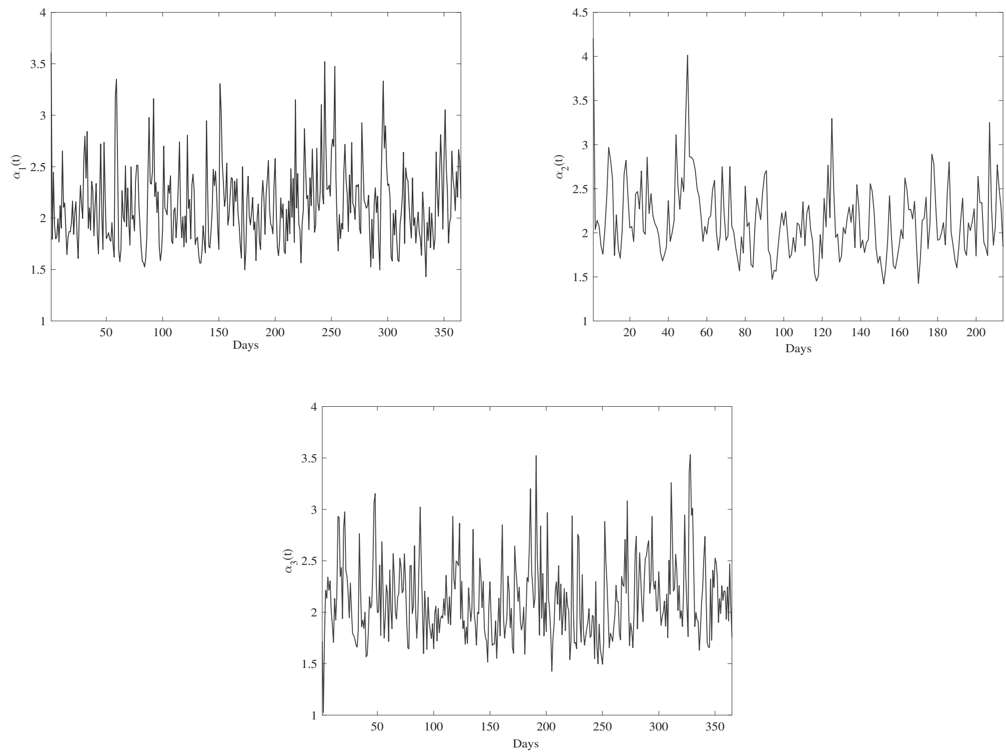

Figure 7: Simulated paths of stochastic speed of mean reversion. Top left: $\alpha_1(t)$. Top right: $\alpha_2(t)$. Buttom: $\alpha_3(t)$.

Next, we compute the eigenvalues of the matrix $A(t)$ at each time t. We found that real parts of eigenvalues of $A(t)$ are negative, which indicate that the process CAR(3) process with stochastic speed of mean reversion satisfied the stationary condition.

To simulate $\mathbf{X}(t)$, we discretize (10) with $L = B$ in the form

$$\mathbf{X}(t+1) = \mathbf{X}(t)e^{A(t)} + e^{A(t)}\mathbf{e}_p\sigma\Delta B(t). \tag{22}$$

The simulation of $\mathbf{X}(t)$ for $t = 1, ..., 365$ days started with $X(0) = (0, 0, 0)'$. From (2), we later obtain $Y(t)$. Figure 8 shows the simulated CAR(3) process with stochastic speed of mean reversion. By adding Y to the seasonal mean values, we get the simulated DATs as in Figure 9.

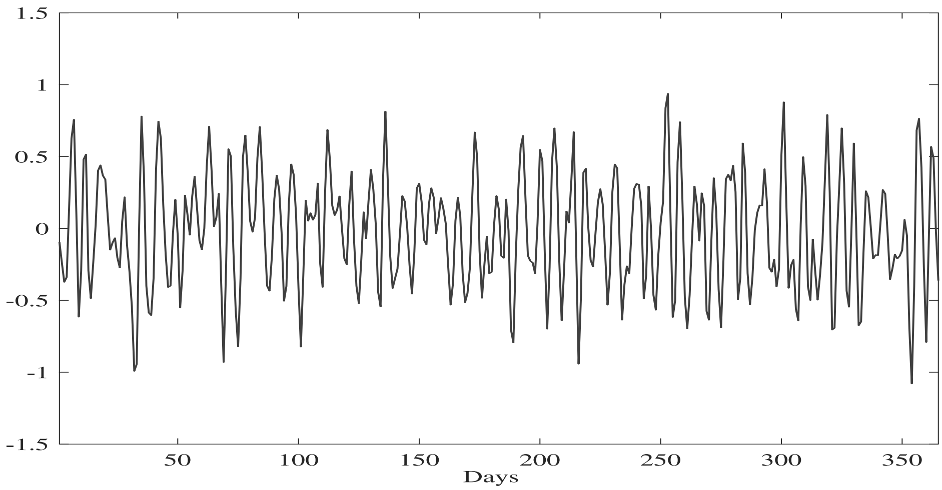

Figure 8: CAR(3) process with stochastic speed of mean reversion.

The simulated temperatures are for 365 days starting from 1 June 2016 until 31 May 2017. We can see DATs moves downwards for the first half of the period, since Kota Bharu is entering monsoon season starting in November where the ambient temperature is low. For the second half of the simulation period, the temperature is going upwards where the monsoon end and sunny season starts. The minimum and maximum temperatures for this simulation period is 25.4 and 29 respectively. We can see that the daily average temperature well simulated by using the CARMA process with stochastic speed of mean reversion, $Y(t)$.

5 Pricing of Temperature Index Insurance

We price the insurance contract based on our simulated $T(t)$ from CAR(3) process with stochastic speed of mean reversion. Suppose that the contract is entered at any time $t \leq \tau_1$ with the coverage period $[\tau_1, \tau_2]$. The coverage period can be monthly, quarterly, semi-annually, or annually. This insurance contract is based on the cumulative average temperature (CAT) index which are measured as sum of DATs during the coverage period, given by

$$\text{CAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T(t)dt.$$

The insured pays the premium at time $t \leq \tau_1$ and will receive an amount of $\text{CAT}(\tau_1, \tau_2)$ at time τ_2. To obtain the price of the insurance, we multiply the average of $\text{CAT}(\tau_1, \tau_2)$ with the discounting factor $\exp(-r(\tau_2 - t))$, where $r > 0$ is the risk-free interest rate to get the present value,

$$P(t, \tau_1, \tau_2) = \exp(-r(\tau_2 - t))\mathbb{E}[\text{CAT}(\tau_1, \tau_2)|\mathcal{F}_t].$$

Suppose we have the temperature insurance with coverage period starting from 1 June 2016 until 30 June 2016. Let the contract is engaged in May 2016. We start the simulation of DATs from 1

May 2016 until 30 June 2016 for 1000 times. Then, we compute the CAT index for each simulation and discount their average to get the present value. The same procedures are repeated for 2 May 2016, the next day until 31 May 2016. We set the discounting rate r equals 0.00014 corresponds to 5% annual interest rate. Figure 9 presents the movement of the price P for the contract in June. The lowest price is RM844.53 at 6 May 2016 and the highest price quoted on 25 May 2016 about RM856.08.

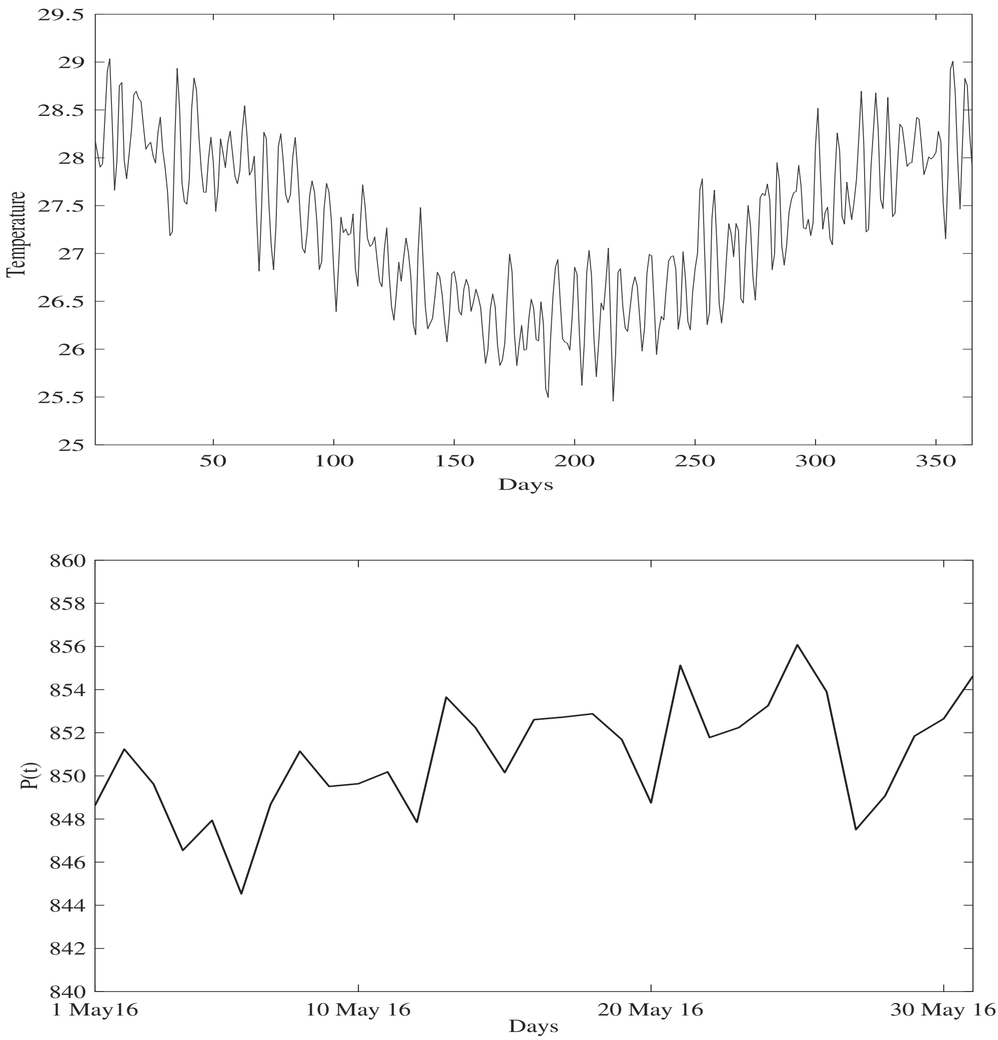

Figure 9: Time series of (top) simulation of DATs for 1 year and (bottom) the movement of the price P for contract in June.

6 Conclusion

We have proposed the CARMA processes with stochastic speed of mean reversion where we allow the mean reversion rates in matrix A to behave stochastically. Their dynamics are explained using the stochastic process of OU type. With the new representation of the stochastic differential equation for the CARMA processes and the integrability of matrix A, we obtained the explicit solution from Ito formula. We also provided the solution for the differential equation of the speed of mean reversion and found that by restricting the driving factor of the OU process for the mean reversion rates dynamics to be subordinator, the stationary property of the process holds.

Our analysis on the Kota Bharu, Malaysia temperatures data has revealed that the CAR model of order 3 is appropriate in explaining the temperature evolution. However, the assumption on normally distributed residuals has been rejected due to the mass concentration of residuals at the center of the distribution and heavy tails. We have fitted the residuals with NIG distribution which looks fit very well. Hence, we assume that the increments of the CAR(3) process are NIG distributed.

We have simulated the temperature dynamics based on the CAR(3) model with NIG distributed errors. In order to simulate the temperature, we first simulate the stochastic speed of mean reversions for certain time period. We have appealed the OU process for the mean reversion rate which later being used for simulating the CAR(3) process. Finally, we simulate the temperature afterwards. From the simulated temperature, we price the temperature index insurance which is computed using the CAT index. Since the index is based on the cumulative temperatures, the price of the insurance looks expensive. From the policyholder perspectives, such price may not be so attractive, but it is reasonable from the insurance companies point of view since they have all the informations to cover their temperature risk.

We may extend the study by considering a stochastic process for volatility dynamics together with the stochastic speed of mean reversion in the OU process. The stochastic volatility is justifiable with regards to the volatility clustering and fat tails in the residual distribution of many stationary time series. But, it is a delicate task to ensure that stationary property of the OU process still hold when both parameters are considered as stochastic. It is also a challenge to provide analytical solution of a new stochastic differential equation for such process. These questions are left for future research.

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Conflicts of Interest The authors declare no conflict of interest.

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